

Generalized Derivations of Lie triple systems *

Jia Zhou¹, Liangyun Chen², Yao Ma²

¹School of Information Technology, Jilin Agricultural University,
Changchun, 130118, CHINA

²School of Mathematics and Statistics, Northeast Normal University,
Changchun, 130024, CHINA

Abstract

In this paper, we present some basic properties concerning the derivation algebra $\text{Der}(T)$, the quasiderivation algebra $\text{QDer}(T)$ and the generalized derivation algebra $\text{GDer}(T)$ of a Lie triple system T , with the relationship $\text{Der}(T) \subseteq \text{QDer}(T) \subseteq \text{GDer}(T) \subseteq \text{End}(T)$. Furthermore, we completely determine those Lie triple systems T with condition $\text{QDer}(T) = \text{End}(T)$. We also show that the quasiderivations of T can be embedded as derivations in a larger Lie triple system.

Key words: Generalized derivations; Quasiderivations; Centroids.

MSC(2010): 16W25, 17B40

§0 Introduction

Lie triple systems arose initially in Cartan's study of Riemannian geometry. Jacobson [8] first introduced them in connection with problems from Jordan theory and quantum mechanics, viewing Lie triple systems as subspaces of Lie algebras that are closed related to the ternary product. Lister gave the structure theory of Lie triple systems of characteristic 0 in [14]. Hopkin introduced the concepts of nilpotent ideals and the nil-radical of Lie triple systems, she successfully generalized Engel's theorem to Lie triple systems in characteristic zero [7]. More recently, Lie triple systems have been connected with the study of the Yang-Baxter equations [10].

As is well known, derivation and generalized derivation algebras are very important subjects in the research of Lie algebras. In the study of Levi factors in derivation algebras of nilpotent Lie algebras, the generalized derivations, quasiderivations, centroids and quasicentroids play key roles [1]. In [15], Melville dealt particularly with the centroids of nilpotent Lie algebras. The most important and systematic research

*Supported by NNSF of China (No. 11171055 and No. 11471090), NSF of Jilin province (No. 201115006). Corresponding author (L. Chen): chenly640@nenu.edu.cn

on the generalized derivation algebras of a Lie algebra and their subalgebras was due to Leger and Luks. In [11], some nice properties of the quasiderivation algebras and of the centroids have been obtained. In particular, they investigated the structure of the generalized derivation algebras and characterized the Lie algebras satisfying certain conditions. Meanwhile, they also pointed that there exist some connections between quasiderivations and cohomology of Lie algebras. For the generalized derivation algebras of more general nonassociative algebras, the readers will be referred to the papers [2–6, 9, 15, 18].

In this paper, we generalize some beautiful results in [11] to Lie triple system. In particular, we seek to understand the structure of the generalized derivation algebras of a Lie triple system or conversely, we want to characterize the Lie triple systems for which the generalized derivation algebras or their Lie subalgebras satisfy some special conditions.

This paper is organized as follows. Section 2 contains some elementary observations about generalized derivations, quasiderivations, centroids and quasiceintroids, some of which are technical results to be used in the sequel. In Section 3, we characterize completely those Lie triple systems T for which $\text{QDer}(T) = \text{End}(T)$. Such Lie triple systems include two-dimensional simple Lie triple systems and all the commutative Lie triple systems. Section 4 is devoted to showing that the quasiderivations of a Lie triple system can be embedded as derivations in a larger Lie triple system \check{T} . Moreover, if the center of T is zero, we obtain a semidirect decomposition of $\text{Der}(\check{T})$.

§1 Preliminaries

Definition 1.1 [16] *A Lie triple system is a pair $(T, [\cdot, \cdot, \cdot])$ consisting of a vector space T over a field \mathbb{F} , a trilinear multiplication $[\cdot, \cdot, \cdot] : T \times T \times T \rightarrow T$ such that for all $x, y, z, u, v \in T$,*

$$\begin{aligned} [x, x, z] &= 0, \\ [x, y, z] + [y, z, x] + [z, x, y] &= 0, \\ [x, y, [z, u, v]] &= [[x, y, z], u, v] + [z, [x, y, u], v] + [z, u, [x, y, v]]. \end{aligned}$$

$\text{End}(T)$ denotes the set consists of all linear maps of T . Obviously, $\text{End}(T)$ is a Lie algebra over \mathbb{F} with the bracket $[D_1, D_2] = D_1 D_2 - D_2 D_1$, for all $D_1, D_2 \in \text{End}(T)$.

Definition 1.2 [17] *Let $(T, [\cdot, \cdot, \cdot])$ be a Lie triple system. A linear map $D : T \rightarrow T$ is said to be a derivation of T if it satisfies*

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = D([x, y, z]),$$

$\forall x, y, z \in T$.

We denote the set of all derivations by $\text{Der}(T)$, then $\text{Der}(T)$ provided with the commutator is a subalgebra of $\text{End}(T)$ and is called the derivation algebra of T .

Definition 1.3 *$D \in \text{End}(T)$ is said to be a generalized derivation of T , if there exist $D', D'', D''' \in \text{End}(T)$ such that*

$$[D(x), y, z] + [x, D'(y), z] + [x, y, D''(z)] = D'''([x, y, z]), \quad (1.1)$$

for all $x, y, z \in T$.

Definition 1.4 $D \in \text{End}(T)$ is said to be a quasiderivation, if there exists $D' \in \text{End}(T)$ such that

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = D'([x, y, z]), \quad (1.2)$$

for all $x, y, z \in T$.

Denote by $\text{GDer}(T)$ and $\text{QDer}(T)$ the sets of generalized derivations and quasiderivations, respectively.

Definition 1.5 [13] If $\text{C}(T) = \{D \in \text{End}(T) \mid [D(x), y, z] = [x, D(y), z] = [x, y, D(z)] = D([x, y, z])\}$ for all $x, y, z \in T$, then $\text{C}(T)$ is called a centroid of T .

Definition 1.6 If $\text{QC}(T) = \{D \in \text{End}(T) \mid [D(x), y, z] = [x, D(y), z] = [x, y, D(z)]\}$ for all $x, y, z \in T$, then $\text{QC}(T)$ is called a quasicentroid of T .

Definition 1.7 If $\text{ZDer}(T) = \{D \in \text{End}(T) \mid [D(x), y, z] = D([x, y, z]) = 0\}$ for all $x, y, z \in T$, then $\text{ZDer}(T)$ is called a central derivation of T .

It is easy to verify that

$$\text{ZDer}(T) \subseteq \text{Der}(T) \subseteq \text{QDer}(T) \subseteq \text{GDer}(T) \subseteq \text{End}(T).$$

Definition 1.8 [13] T is a Lie triple system and I is a non-empty subset of T . We call $Z_T(I) = \{x \in T \mid [x, a, y] = [y, a, x] = 0, \forall a \in I, y \in T\}$ the centralizer of I in T . In particular, $Z_T(T) = \{x \in T \mid [x, y, z] = 0, \forall y, z \in T\}$ is the center of T , denoted by $Z(T)$.

§2 Generalized derivation algebras and their subalgebras

First, we give some basic properties of center derivation algebra, quasiderivation algebra and the generalized derivation algebra of a Lie triple system.

Proposition 2.1 Let T be a Lie triple system. Then the following statements hold:

- (1) $\text{GDer}(T)$, $\text{QGer}(T)$ and $\text{C}(T)$ are subalgebras of $\text{End}(T)$.
- (2) $\text{ZDer}(T)$ is an ideal of $\text{Der}(T)$.

Proof. (1) Assume that $D_1, D_2 \in \text{GDer}(T)$. For all $x, y, z \in T$, we have

$$\begin{aligned} [D_1 D_2(x), y, z] &= D_1''' D_2'''([x, y, z]) - D_1'''[x, D_2'(y), z] - D_1'''[x, y, D_2''(z)] \\ &\quad - [D_2(x), D_1'(y), z] - [D_2(x), y, D_1''(z)] \\ &= D_1''' D_2'''([x, y, z]) - [D_1(x), D_2'(y), z] - [x, D_1' D_2'(y), z] \\ &\quad - [x, D_2'(y), D_1''(z)] - [D_1(x), y, D_2''(z)] - [x, D_1'(y), D_2''(z)] \\ &\quad - [x, y, D_1'' D_2''(z)] - [D_2(x), D_1'(y), z] - [D_2(x), y, D_1''(z)], \end{aligned}$$

and

$$\begin{aligned} [D_2 D_1(x), y, z] &= D_2''' D_1'''([x, y, z]) - D_2'''[x, D_1'(y), z] - D_2'''[x, y, D_1''(z)] \\ &\quad - [D_1(x), D_2'(y), z] - [D_1(x), y, D_2''(z)] \\ &= D_2''' D_1'''([x, y, z]) - [D_2(x), D_1'(y), z] - [x, D_2' D_1'(y), z] \\ &\quad - [x, D_1'(y), D_2''(z)] - [D_2(x), y, D_1''(z)] - [x, D_2'(y), D_1''(z)] \\ &\quad - [x, y, D_2'' D_1''(z)] - [D_1(x), D_2'(y), z] - [D_1(x), y, D_2''(z)] \end{aligned}$$

Thus for all $x, y, z \in T$, we have

$$[[D_1, D_2](x), y, z] = [D_1''', D_2''']([x, y, z]) - [x, y, [D_1'', D_2''](z)] - [x, [D_1', D_2'](y), z].$$

From the definition of generalized derivation, one gets $[D_1, D_2] \in \text{GDer}(T)$, so $\text{GDer}(T)$ is a subalgebra of $\text{End}(T)$.

Similarly, $\text{QGer}(T)$ is a subalgebra of $\text{End}(T)$.

Assume that $D_1, D_2 \in \text{C}(T)$. $\forall x, y, z \in T$, note that

$$\begin{aligned} [[D_1, D_2](x), y, z] &= [D_1 D_2(x), y, z] - [D_2 D_1(x), y, z] \\ &= D_1([D_2(x), y, z]) - D_2([D_1(x), y, z]) \\ &= D_1 D_2([x, y, z]) - D_2 D_1([x, y, z]) \\ &= [D_1, D_2]([x, y, z]). \end{aligned}$$

Similarly,

$$[x, [D_1, D_2](y), z] = [D_1, D_2]([x, y, z]) = [x, y, [D_1, D_2](z)].$$

Then $[D_1, D_2] \in \text{C}(T)$, $\text{C}(T)$ is a subalgebra of $\text{End}(T)$.

(2) Assume that $D_1 \in \text{ZDer}(T)$, $D_2 \in \text{Der}(T)$. For all $x, y, z \in T$, we have

$$[[D_1, D_2]([x, y, z])] = D_1 D_2([x, y, z]) - D_2 D_1([x, y, z]) = 0,$$

and

$$\begin{aligned} [[D_1, D_2](x), y, z] &= [(D_1 D_2 - D_2 D_1)(x), y, z] \\ &= D_1([D_2(x), y, z]) - [D_1(x), D_2(y), z] = 0. \end{aligned}$$

Then $[D_1, D_2] \in \text{ZDer}(T)$ and $\text{ZDer}(T)$ is an ideal of $\text{Der}(T)$. □

Lemma 2.2 *Let T be a Lie triple system. Then*

- (1) $[\text{Der}(T), \text{C}(T)] \subseteq \text{C}(T)$;
- (2) $[\text{QDer}(T), \text{QC}(T)] \subseteq \text{QC}(T)$;
- (3) $\text{C}(T) \cdot \text{Der}(T) \subseteq \text{Der}(T)$;
- (4) $\text{C}(T) \subseteq \text{QDer}(T)$;
- (5) $[\text{QC}(T), \text{QC}(T)] \subseteq \text{QDer}(T)$;
- (6) $\text{QDer}(T) + \text{QC}(T) \subseteq \text{GDer}(T)$.

Proof. (1) Assume that $D_1 \in \text{Der}(T)$, $D_2 \in \text{C}(T)$. For all $x, y, z \in T$, we have

$$\begin{aligned} [D_1 D_2(x), y, z] &= D_1([D_2(x), y, z]) - [D_2(x), D_1(y), z] - [D_2(x), y, D_1(z)] \\ &= D_1 D_2([x, y, z]) - [x, D_2 D_1(y), z] - [x, y, D_2 D_1(z)], \end{aligned}$$

and

$$\begin{aligned} [D_2 D_1(x), y, z] &= D_2(D_1([x, y, z]) - [x, D_1(y), z] - [x, y, D_1(z)]) \\ &= D_2 D_1([x, y, z]) - [x, D_2 D_1(y), z] - [x, y, D_2 D_1(z)]. \end{aligned}$$

Hence,

$$[[D_1, D_2](x), y, z] = D_1 D_2([x, y, z]) - D_2 D_1([x, y, z]) = [D_1, D_2]([x, y, z]).$$

Similarly,

$$[[D_1, D_2](x), y, z] = [x, [D_1, D_2](y), z] = [x, y, [D_1, D_2](z)].$$

Thus, $[D_1, D_2] \in C(T)$ and we get $[\text{Der}(T), C(T)] \subseteq C(T)$.

(2) Similar to the proof of (1).

(3) Assume that $D_1 \in C(T)$, $D_2 \in \text{Der}(T)$. For all $x, y, z \in T$, we have

$$\begin{aligned} D_1 D_2[x, y, z] &= D_1([D_2(x), y, z] + [x, D_2(y), z] + [x, y, D_2(z)]) \\ &= [D_1 D_2(x), y, z] + [x, D_1 D_2(y), z] + [x, y, D_1 D_2(z)]. \end{aligned}$$

So we have $D_1 D_2 \in \text{Der}(T)$.

(4) Assume that $D \in \text{QC}(T)$. For all $x, y, z \in T$, we have

$$[D(x), y, z] = [x, D(y), z] = [x, y, D(z)].$$

Hence,

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = 3D[x, y, z].$$

Therefore, $D \in \text{QDer}(T)$ since $D' = 3D \in C(T) \subseteq \text{End}(T)$.

(5) Assume that $D_1, D_2 \in \text{QC}(T)$. For all $x, y, z \in T$, we have

$$\begin{aligned} & [[D_1, D_2](x), y, z] + [x, [D_1, D_2](y), z] + [x, y, [D_1, D_2](z)] \\ &= [D_1 D_2(x), y, z] + [x, D_1 D_2(y), z] + [x, y, D_1 D_2(z)] - [D_2 D_1(x), y, z] \\ & \quad - [x, D_2 D_1(y), z] - [x, y, D_2 D_1(z)]. \end{aligned}$$

And

$$\begin{aligned} [D_1 D_2(x), y, z] &= [D_2(x), D_1(y), z] = [x, D_2 D_1(y), z], \\ [D_1 D_2(x), y, z] &= [D_2(x), y, D_1(z)] = [x, y, D_2 D_1(z)]. \end{aligned}$$

Hence,

$$[[D_1, D_2](x), y, z] + [x, [D_1, D_2](y), z] + [x, y, [D_1, D_2](z)] = 0,$$

i.e. $[D_1, D_2] \in \text{QDer}(T)$.

(6) It is obvious. □

Lemma 2.3 [17] *If T is a Lie triple system, I is an ideal of T , then $Z_T(I)$ is also an ideal of T . Moreover, $Z(T) = Z_T(T)$, $Z(I) = Z_I(I)$ are ideals of T .*

Lemma 2.4 [17] *Let the Lie triple system T be decomposed into the direct sum of two ideals, i.e. $T = A \oplus B$. Then we have*

- (1) $Z(T) = Z(A) \oplus Z(B)$.
- (2) If $Z(T) = 0$, then $\text{Der}(T) = \text{Der}(A) \oplus \text{Der}(B)$.

Proposition 2.5 *If the Lie triple system T can be decomposed into the direct sum of two ideals, i.e. $T = A \oplus B$ and $Z(T) = 0$, then we have*

- (1) $\text{GDer}(T) = \text{GDer}(A) \oplus \text{GDer}(B)$;
- (2) $\text{QDer}(T) = \text{QDer}(A) \oplus \text{QDer}(B)$;
- (3) $C(T) = C(A) \oplus C(B)$;
- (4) $\text{QC}(T) = \text{QC}(A) \oplus \text{QC}(B)$.

Proof. (1) For $D' \in \text{GDer}(A)$, extend it to a linear transformation on T by setting $D'(a+b) = D'(a)$, $\forall a \in A, b \in B$. Obviously, $D' \in \text{GDer}(T)$ and $\text{GDer}(A) \subseteq \text{GDer}(T)$. Similarly, $\text{GDer}(B) \subseteq \text{GDer}(T)$. Let $a \in A, b_1, b_2 \in B$ and $D \in \text{Der}(T)$. Then

$$\begin{aligned} [D(a), b_1, b_2] &= D([a, b_1, b_2]) - [a, D(b_1), b_2] - [a, b_1, D(b_2)] \\ &= -[a, D(b_1), b_2] - [a, b_1, D(b_2)] \in A \cap B = \{0\}. \end{aligned}$$

Suppose $D(a) = a' + b'$, where $a \in A, b \in B$, then

$$0 = [D(a), b_1, b_2] = [a', b_1, b_2] + [b', b_1, b_2].$$

So $[b', b_1, b_2] = 0$ and $b' \in Z(B)$. Since $Z(T) = Z(A) \oplus Z(B)$, $b' = 0$. Hence $D(a) = a' \in A$. Therefore $D(A) \subseteq A$. Similarly, $D(B) \subseteq B$.

Let $D \in \text{Der}(T)$ and $x = a + b \in A + B$, where $a \in A, b \in B$. Define $E, F \in \text{End}(T)$ by $E(a+b) = D(a), F(a+b) = D(b)$, then $E \in \text{Der}(A), F \in \text{Der}(B)$. Hence $D = E + F \in \text{Der}(A) + \text{Der}(B)$. Since $\text{Der}(A) \cap \text{Der}(B) = \{0\}$, $\text{GDer}(T) = \text{GDer}(A) \dot{+} \text{GDer}(B)$ as a vector space.

Let $E \in \text{Der}(A), F \in \text{Der}(B)$ and $b \in B$. Then $[E, D] = (ED - DE)(b) = 0$. Hence $[E, D] \in \text{Der}(A)$ and $\text{Der}(A) \triangleleft \text{Der}(T)$. Similarly, $\text{Der}(B) \triangleleft \text{Der}(T)$.

(2), (3), (4) Similar to the proof of (1). \square

Proposition 2.6 *If T is a Lie triple system, then $\text{QC}(T) + [\text{QC}(T), \text{QC}(T)]$ is a subalgebra of $\text{GDer}(T)$.*

Proof. By the conclusions of Lemma 2.2 (5) and (6), we have

$$\text{QC}(T) + [\text{QC}(T), \text{QC}(T)] \subseteq \text{GDer}(T)$$

and

$$\begin{aligned} &[\text{QC}(T) + [\text{QC}(T), \text{QC}(T)], \text{QC}(T) + [\text{QC}(T), \text{QC}(T)]] \\ &\subseteq [\text{QC}(T) + \text{QDer}(T), \text{QC}(T) + [\text{QC}(T), \text{QC}(T)]] \\ &\subseteq [\text{QC}(T), \text{QC}(T)] + [\text{QC}(T), [\text{QC}(T), \text{QC}(T)]] + [\text{QDer}(T), \text{QC}(T)] \\ &\quad + [\text{QDer}(T), [\text{QC}(T), \text{QC}(T)]]. \end{aligned}$$

It is easy to verify $[\text{QDer}(L), [\text{QC}(L), \text{QC}(L)]] \subseteq [\text{QC}(L), \text{QC}(L)]$ by the Jacobi identity of Lie algebra. Thus

$$[\text{QC}(L) + [\text{QC}(L), \text{QC}(L)], \text{QC}(L) + [\text{QC}(L), \text{QC}(L)]] \subseteq \text{QC}(L) + [\text{QC}(L), \text{QC}(L)].$$

Theorem 2.7 *If T is a Lie triple system, then $[\text{C}(T), \text{QC}(T)] \subseteq \text{End}(T, Z(T))$. Moreover, if $Z(T) = \{0\}$, then $[\text{C}(T), \text{QC}(T)] = \{0\}$.*

Proof. Assume that $D_1 \in \text{C}(T), D_2 \in \text{QC}(T)$ and for all $x, y, z \in T$, we have

$$\begin{aligned} [[D_1, D_2](x), y, z] &= [D_1 D_2(x), y, z] - [D_2 D_1(x), y, z] \\ &= D_1([D_2(x), y, z]) - [D_1(x), D_2(y), z] \\ &= D_1([D_2(x), y, z]) - [x, D_2(y), z] = 0. \end{aligned}$$

Hence $[D_1, D_2](x) \in Z(T)$ and $[D_1, D_2] \in \text{End}(T, Z(T))$ as desired. Furthermore, if $Z(T) = \{0\}$, it is clear that $[C(T), \text{QC}(T)] = \{0\}$. \square

Definition 2.8 [19] *Let L be an algebra over \mathbb{F} ($\text{char } \mathbb{F} \neq 2$), if the multiplication satisfies the following identities:*

$$\begin{aligned} x \cdot y &= y \cdot x, \\ (((x \cdot y) \cdot w) \cdot z - (x \cdot y) \cdot (w \cdot z)) &+ (((y \cdot z) \cdot w) \cdot x - (y \cdot z) \cdot (w \cdot x)) \\ &+ (((z \cdot x) \cdot w) \cdot y - (z \cdot x) \cdot (w \cdot y)) = 0, \end{aligned}$$

for all $x, y, z, w \in L$, then we call L a Jordan algebra.

Proposition 2.9 [19] *Let T be a Lie triple system over \mathbb{F} ($\text{char } \mathbb{F} \neq 2$), with the operation $D_1 \bullet D_2 = D_1 D_2 + D_2 D_1$, for all elements $D_1, D_2 \in \text{End}(T)$. Then the pair $(\text{End}(T), \bullet)$ is a Jordan algebra.*

Corollary 2.10 *Let T be a Lie triple system over \mathbb{F} ($\text{char } \mathbb{F} \neq 2$), with the operation $D_1 \bullet D_2 = D_1 D_2 + D_2 D_1$, for all elements $D_1, D_2 \in \text{QC}(T)$. Then $(\text{QC}(T), \bullet)$ is a Jordan algebra.*

Proof. We need only to show that $D_1 \bullet D_2 \in \text{QC}(T)$, for all $D_1, D_2 \in \text{QC}(T)$, we have

$$\begin{aligned} [D_1 \bullet D_2(x), y, z] &= [D_1 D_2(x), y, z] + [D_2 D_1(x), y, z] \\ &= [D_2(x), D_1(y), z] + [D_1(x), D_2(y), z] \\ &= [x, D_2 D_1(y), z] + [x, D_1 D_2(y), z] \\ &= [x, D_1 \bullet D_2(y), z]. \end{aligned}$$

Similarly, $[D_1 \bullet D_2(x), y, z] = [x, y, D_1 \bullet D_2(z)]$. Then $D_1 \bullet D_2 \in \text{QC}(T)$ and $\text{QC}(T)$ is a Jordan algebra.

Theorem 2.11 *If T is a Lie triple system over \mathbb{F} , then we have*

- (1) *If $\text{char } \mathbb{F} \neq 2$, then $\text{QC}(T)$ is a Lie algebra with $[D_1, D_2] = D_1 D_2 - D_2 D_1$ if and only if $\text{QC}(T)$ is also an associative algebra with respect to composition.*
- (2) *If $\text{char } \mathbb{F} \neq 3$ and $Z(T) = \{0\}$, then $\text{QC}(T)$ is a Lie algebra if and only if $[\text{QC}(T), \text{QC}(T)] = 0$.*

Proof. (1) (\Leftarrow) For all $D_1, D_2 \in \text{QC}(T)$, we have $D_1 D_2 \in \text{QC}(T)$ and $D_2 D_1 \in \text{QC}(T)$, so $[D_1, D_2] = D_1 D_2 - D_2 D_1 \in \text{QC}(T)$. Hence, $\text{QC}(T)$ is a Lie algebra.

(\Rightarrow) Note that $D_1 D_2 = D_1 \bullet D_2 + \frac{[D_1, D_2]}{2}$ and by Corollary 2.10, we have $D_1 \bullet D_2 \in \text{QC}(T)$, $[D_1, D_2] \in \text{QC}(T)$. It follows that $D_1 D_2 \in \text{QC}(T)$ as desired.

(2) (\Rightarrow) Assume that $D_1, D_2 \in \text{QC}(T)$. For all $x, y, z \in T$, $\text{QC}(T)$ is a Lie algebra, so $[D_1, D_2] \in \text{QC}(T)$, then

$$[[D_1, D_2](x), y, z] = [x, [D_1, D_2](y), z] = [x, y, [D_1, D_2](z)].$$

From the proof of Lemma 2.2 (5), we have

$$[[D_1, D_2](x), y, z] = -[x, [D_1, D_2](y), z] - [x, y, [D_1, D_2](z)].$$

Hence $3[[D_1, D_2](x), y, z] = 0$. We have $[[D_1, D_2](x), y, z] = 0$, i.e. $[D_1, D_2] = 0$ since $\text{char } \mathbb{F} \neq 3$.

(\Leftarrow) It is clear. \square

Lemma 2.12 *Let V be a linear space and $\mathcal{A} : V \rightarrow V$ a linear map. $f(x)$ denotes the minimal polynomial of f . If x^2 does not divide $f(x)$, then $V = \text{Ker}(\mathcal{A}) \dot{+} \text{Im}(\mathcal{A})$.*

Proof. Obviously $\dim(V) = \dim(\text{Ker}(\mathcal{A})) + \dim(\text{Im}(\mathcal{A}))$ because \mathcal{A} is a linear map. x^2 does not divide $f(x)$ means that $f(x) = x^2g(x) + ax + b$, $a \neq 0$ or $b \neq 0$.

Case 1: If $b \neq 0$, then $f(\mathcal{A}) = \mathcal{A}^2g(\mathcal{A}) + a\mathcal{A} + b\text{id} = 0$, so $\mathcal{A}(\mathcal{A}g(\mathcal{A}) + a\text{id}) = -b\text{id}$ and \mathcal{A} is invertible. Hence $\text{Ker}(\mathcal{A}) = \{0\}$.

Case 2: If $b = 0$, that means $a \neq 0$, $f(\mathcal{A}) = \mathcal{A}^2g(\mathcal{A}) + a\mathcal{A}$. Here we only prove $\text{Ker}(\mathcal{A}) \cap \text{Im}(\mathcal{A}) = \{0\}$. Indeed, $\forall x \in \text{Ker}(\mathcal{A}) \cap \text{Im}(\mathcal{A})$, $\mathcal{A}(x) = 0$ and there exists $x' \in V$ such that $\mathcal{A}(x') = x$. So $f(\mathcal{A})(x') = \mathcal{A}^2g(\mathcal{A})(x') + a\mathcal{A}(x')$, which means $a\mathcal{A}(x') = ax = 0$. Hence $x = 0$ since $a \neq 0$. \square

Proposition 2.13 *Let T be a Lie triple system, $D \in C(T)$. Then*

- (1) $\text{Ker}(D)$ and $\text{Im}(D)$ are ideals in T .
- (2) If T is indecomposable, $D \in C(T)$ and $D \neq 0$. Suppose x^2 does not divide the minimal polynomial of D , then D is invertible.
- (3) If T is indecomposable and $C(T)$ consists of semisimple elements, then $C(T)$ is a field.

Proof. (1) Since $D \in C(T)$, for all $x \in \text{Ker}(D)$, $y, z \in T$, one gets $D[x, y, z] = [D(x), y, z] = 0$, that means $[x, y, z] \in \text{Ker}(D)$.

Meanwhile, for all $x \in \text{Im}(D)$, there exists $x' \in T$, such that $x = D(x')$. So $[x, y, z] = [D(x'), y, z] = D[x', y, z] \in \text{Im}(D)$.

(2) From Lemma 2.12 and (1) there is an ideal sum $T = \text{Ker}(D) \oplus \text{Im}(D)$. Since T is indecomposable, one gets $\text{Ker}(D) = 0$ and $\text{Im}(D) = T$, which means D is invertible.

(3) For all semisimple element $D \in C(T)$, since x^2 does not divide the minimal polynomial of D and T is indecomposable, from (2) one gets D is invertible. It is obvious that $\text{id} \in C(T)$. If there exist $D_1 \neq 0$, $D_2 \neq 0$, $D_1, D_2 \in C(T)$ such that $D_1D_2 = 0$, then $D_1 = D_2 = 0$, a contradiction. Hence $C(T)$ has no zero divisor. Obviously, one gets $D_1D_2 = D_2D_1$, $\forall D_1, D_2 \in C(T)$. So $C(T)$ is a field. \square

Lemma 2.14 *Let T be a Lie triple system with $Z(T) = \{0\}$. If $D \in \text{QC}(T)$ and suppose x^2 does not divide the minimal polynomial of D , then $T = \text{Ker}(D) \oplus \text{Im}(D)$.*

Proof. From Lemma 2.12, there is a vector space direct sum $T = \text{Ker}(D) \dot{+} \text{Im}(D)$. Obviously, $[\text{Ker}(D), D(T), T] = [D(\text{Ker}(D)), T, T] = 0$ and $[T, D(T), \text{Ker}(D)] = [T, T, D(\text{Ker}(D))] = 0$, so $\text{Ker}(D) \subseteq Z_T(\text{Im}(D))$, $\text{Im}(D) \subseteq Z_T(\text{Ker}(D))$. Since $Z_T(\text{Im}(D)) \cap Z_T(\text{Ker}(D)) = Z(T) = \{0\}$, we must have $\text{Ker}(D) = Z_T(\text{Im}(D))$, $\text{Im}(D) = Z_T(\text{Ker}(D))$.

It is easy to get $[[\text{Ker}(D), T, T], \text{Im}(D), T] = [T, \text{Im}(D), [\text{Ker}(D), T, T]] = 0$, which means $[\text{Ker}(D), T, T] \subseteq Z_T(\text{Im}(D)) = \text{Ker}(D)$.

Also $[[\text{Im}(D), T, T], \text{Ker}(D), T] = [T, \text{Ker}(D), [\text{Im}(D), T, T]] = 0$, $[\text{Im}(D), T, T] \subseteq Z_T(\text{Ker}(D)) = \text{Im}(D)$. So $\text{Ker}(D)$ and $\text{Im}(D)$ are ideals. \square

Corollary 2.15 *Let $(T, [\cdot, \cdot, \cdot])$ be a indecomposable Lie triple system over an alge-*

braically field \mathbb{F} and $Z(T) = 0$. $D \in \text{QC}(T)$ is semisimple, then $D \in Z_{\text{C}(T)}(\text{GDer}(T))$.

Proof. Let $D \in \text{QC}(T)$, D has an eigenvalue λ since \mathbb{F} is an algebraically field. We denote the corresponding eigenspace by $E_\lambda(D)$, it is easy to get $(D - \lambda \text{id}) \in \text{QC}(T)$ and $\text{Ker}(D - \lambda \text{id}) = E_\lambda(D) \neq 0$. From Lemma 2.14 and D is a semisimple element, $\text{Ker}(D - \lambda \text{id})$ is an ideal of T , so $\text{Ker}(D - \lambda \text{id}) = T$. That is $D = \lambda \text{id} \in \text{C}(T)$ and $[D, \text{GDer}(T)] = 0$. \square

§3 Lie triple systems with $\text{QDer}(T) = \text{End}(T)$

Let T be a Lie triple system over \mathbb{F} . We define a linear map $\phi : T \otimes T \otimes T \rightarrow T, x \otimes y \otimes z \mapsto [x, y, z]$. Define $\text{Ker}(\phi) := \{\sum x \otimes y \otimes z \in T \otimes T \otimes T \mid x, y, z \in T, \sum [x, y, z] = 0\}$, then it is easy to see that $\text{Ker}(\phi)$ is a subspace of $T \otimes T \otimes T$.

We define $(T \otimes T \otimes T)^+ := \langle x \otimes y \otimes z + y \otimes x \otimes z \mid x, y, z \in T \rangle$ and $(T \otimes T \otimes T)^- := \langle x \otimes y \otimes z - y \otimes x \otimes z \mid x, y, z \in T \rangle$, then both $(T \otimes T \otimes T)^+$ and $(T \otimes T \otimes T)^-$ are subspaces of $T \otimes T \otimes T$. It is easy to check that

$$T \otimes T \otimes T = (T \otimes T \otimes T)^+ \dot{+} (T \otimes T \otimes T)^- \text{ (direct sum of vector spaces),}$$

and we also have $\dim(T \otimes T \otimes T)^+ = n^2(n+1)/2$ and $\dim(T \otimes T \otimes T)^- = n^2(n-1)/2$, where $\dim(T) = n$.

For all $D \in \text{End}(T)$, we define $D^* \in \text{End}(T \otimes T \otimes T)$ satisfying that

$$D^*(x \otimes y \otimes z) = D(x) \otimes y \otimes z + x \otimes D(y) \otimes z + x \otimes y \otimes D(z), \quad (3.1)$$

for all $x, y, z \in T$.

Lemma 3.1 $D \in \text{QDer}(T)$ if and only if $D^*(\text{Ker}(\phi)) \subseteq \text{Ker}(\phi)$.

Proof. (\Rightarrow) For all $\sum x \otimes y \otimes z \in \text{Ker}(\phi)$, we have $\sum [x, y, z] = 0$. Thus,

$$\begin{aligned} D^*\left(\sum x \otimes y \otimes z\right) &= \sum D^*(x \otimes y \otimes z) \\ &= \sum (D(x) \otimes y \otimes z + x \otimes D(y) \otimes z + x \otimes y \otimes D(z)). \end{aligned}$$

Since $D \in \text{QDer}(T)$, we have

$$\sum ([D(x), y, z] + [x, D(y), z] + [x, y, D(z)]) = \sum D'([x, y, z]) = D'(\sum [x, y, z]) = 0.$$

Hence $D^*(\text{Ker}(\phi)) \subseteq \text{Ker}(\phi)$.

(\Leftarrow) Since $D^*(\text{Ker}(\phi)) \subseteq \text{Ker}(\phi)$, there exists an element $D' \in \text{End}(T)$ such that

$$\phi \circ D^* = D' \circ \phi : T \otimes T \otimes T \rightarrow T,$$

for all $D \in \text{End}(T)$. A direct computation shows that for all $x, y, z \in T$,

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = D'([x, y, z]),$$

that is, $D \in \text{QDer}(T)$. \square

Lemma 3.2 Suppose that $\text{End}(T)$ acts on $T \otimes T \otimes T$ via $D \cdot (x \otimes y \otimes z) = D^*(x \otimes y \otimes z)$ for all $x, y, z \in T$ with D^* as above. Then $(T \otimes T \otimes T)^+$ and $(T \otimes T \otimes T)^-$ are two irreducible $\text{End}(T)$ -modules.

Proof. Here we need only to show that $(T \otimes T \otimes T)^+$ is an irreducible $\text{End}(T)$ -module and the case of $(T \otimes T \otimes T)^-$ is similar. For all $x, y, z \in T$ and $D \in \text{End}(T)$, from Eq.(3.1) we have

$$\begin{aligned}
& D \cdot (x \otimes y \otimes z + y \otimes x \otimes z) \\
&= D \cdot (x \otimes y \otimes z) + D \cdot (y \otimes x \otimes z) \\
&= D(x) \otimes y \otimes z + x \otimes D(y) \otimes z + x \otimes y \otimes D(z) + D(y) \otimes x \otimes z \\
&\quad + y \otimes D(x) \otimes z + y \otimes x \otimes D(z) \\
&= (D(x) \otimes y \otimes z + y \otimes D(x) \otimes z) + (x \otimes y \otimes D(z) + y \otimes x \otimes D(z)) \\
&\quad + (x \otimes D(y) \otimes z + D(y) \otimes x \otimes z) \in (T \otimes T \otimes T)^+.
\end{aligned}$$

Hence $D \cdot (T \otimes T \otimes T)^+ \subseteq (T \otimes T \otimes T)^+$ and it is easy to check that

$$[D_1, D_2] \cdot (x \otimes y \otimes z) = D_1 \cdot (D_2 \cdot (x \otimes y \otimes z)) - D_2 \cdot (D_1 \cdot (x \otimes y \otimes z)).$$

Therefore, $(T \otimes T \otimes T)^+$ is an $\text{End}(T)$ -module.

Suppose that there exists a nonzero $\text{End}(T)$ -submodule V in $(T \otimes T \otimes T)^+$. Choose a nonzero element $\sum(x \otimes y \otimes z + y \otimes x \otimes z)$ in V for some $x, y, z \in T$. A direct computation shows that all elements in $(T \otimes T \otimes T)^+$ are obtainable by repeated application of elements of $\text{End}(T)$ to $\sum(x \otimes y \otimes z + y \otimes x \otimes z)$ and formation of linear combinations. Hence V is $(T \otimes T \otimes T)^+$ itself. Thus, $(T \otimes T \otimes T)^+$ as an $\text{End}(T)$ -module is irreducible. \square

Theorem 3.3 *Let T be a Lie triple system with $[T, T, T] \neq 0$ and $\text{QDer}(T) = \text{End}(T)$. Then T is a two-dimensional simple Lie triple system.*

Proof. We consider the action of $\text{End}(T)$ on $T \otimes T \otimes T$ via $D \cdot (x \otimes y \otimes z) = D^*(x \otimes y \otimes z)$ for all $x, y, z \in T$ with D^* as in Eq. (3.1). By Lemma 3.1, $\text{QDer}(T) = \text{End}(T)$ implies that $\text{End}(T) \cdot \text{Ker}(\phi) \subseteq \text{Ker}(\phi)$. Lemma 3.2 tells us that the only proper subspaces of $T \otimes T \otimes T$, invariant under this action of $\text{End}(T)$, are $(T \otimes T \otimes T)^+$ and $(T \otimes T \otimes T)^-$. Thus we have $\phi : T \otimes T \otimes T \rightarrow T$ with kernel $\{0\}, (T \otimes T \otimes T)^+$ and $(T \otimes T \otimes T)^-$. Using

$$\dim(T) \geq \dim(T \otimes T \otimes T) - \dim(\text{Ker}(\phi)),$$

we have that $n = 1$, if $\text{Ker}(\phi) = \{0\}$ or $(T \otimes T \otimes T)^-$; $n \leq 2$, if $\text{Ker}(\phi) = (T \otimes T \otimes T)^+$.

We discuss the possibilities for n as follows:

(a) If $n = 1$, then T is commutative, which is a contradiction with our assumption.

(b) If $n = 2$, i.e. $\text{Ker}(\phi) = (T \otimes T \otimes T)^+$, hence $\dim(\text{Ker}(\phi)) = 6$ and $\dim([T, T, T]) = 2$, so ϕ must be surjective and we have $[T, T, T] = T$. So that T is the two-dimensional simple Lie triple system (See [12, Chapter 4.3]). \square

On the other hand, the converse of Theorem 3.3 is also valid. One can prove the following theorem.

Theorem 3.5 *If T is a two-dimensional simple Lie triple system or an abelian Lie triple system, then $\text{QDer}(T) = \text{End}(T)$.*

Proof. By [12, Chapter 4.3], we have a basis e_1, e_2 such that $[e_1, e_2, e_1] = -e_1$, $[e_1, e_2, e_2] = e_2$. Then for all $D \in \text{End}(T)$, one gets $D(e_1) = k_1 e_1 + k'_2 e_2$, $k_1, k'_1, k_2, k'_2 \in \mathbb{F}$. It obvious $k' = 0$ since D is a linear map. Similarly, $D(e_2) = k_2 e_2$. So we have

$$[D(e_1), e_2, e_1] + [e_1, D(e_2), e_1] + [e_1, e_2, D(e_1)] = -(2k_1 + k_2)e_1.$$

$$[D(e_1), e_2, e_2] + [e_1, D(e_2), e_2] + [e_1, e_2, D(e_2)] = (k_1 + 2k_2)e_2.$$

Let $D' \in \text{End}(T)$ such that $D'(e_1) = -(2k_1 + k_2)e_1$ and $D'(e_2) = (k_1 + 2k_2)e_2$, thus $D \in \text{QDer}(T)$. \square

§4 The quasiderivations of Lie triple systems

In this section, we will prove that the quasiderivations of T can be embedded as derivations in a larger Lie triple system and obtain a direct sum decomposition of $\text{Der}(T)$ when the center $Z(T)$ is equal to zero.

Proposition 4.1 *Let T be a Lie triple system over \mathbb{F} and t an indeterminate. We define $\check{T} := \{\Sigma(x \otimes t + y \otimes t^3) | x, y \in T\}$. Then \check{T} is a Lie triple system with the operation $[x \otimes t^i, y \otimes t^j, z \otimes t^k] = [x, y, z] \otimes t^{i+j+k}$, for all $x, y, z \in T, i, j, k \in \{1, 3\}$.*

Proof. For all $x, y, z, u, v \in T$ and $i, j, k, m, n \in \{1, 3\}$, we have

$$\begin{aligned} [x \otimes t^i, y \otimes t^j, z \otimes t^k] &= [x, y, z] \otimes t^{i+j+k} \\ &= -[y, x, z] \otimes t^{i+j+k} \\ &= -[y \otimes t^j, x \otimes t^i, z \otimes t^k], \end{aligned}$$

$$\begin{aligned} &[x \otimes t^i, y \otimes t^j, z \otimes t^k] + [y \otimes t^j, z \otimes t^k, x \otimes t^i] + [z \otimes t^k, x \otimes t^i, y \otimes t^j] \\ &= [x, y, z] \otimes t^{i+j+k} + [y, z, x] \otimes t^{i+j+k} + [z, x, y] \otimes t^{i+j+k} \\ &= ([x, y, z] + [y, z, x] + [z, x, y]) \otimes t^{i+j+k} = 0, \end{aligned}$$

and

$$\begin{aligned} &[x \otimes t^i, y \otimes t^j, [z \otimes t^k, u \otimes t^m, v \otimes t^n]] = [x, y, [z, u, v]] \otimes t^{i+j+k+m+n} \\ &= ([x, y, z], u, v] + [z, [x, y, u], v] + [z, u, [x, y, v]]) \otimes t^{i+j+k+m+n} \\ &= [[x \otimes t^i, y \otimes t^j, z \otimes t^k], u \otimes t^m, v \otimes t^n] + [z \otimes t^k, [x \otimes t^i, y \otimes t^j, u \otimes t^m], v \otimes t^n] \\ &\quad + [z \otimes t^k, u \otimes t^m, [x \otimes t^i, y \otimes t^j, v \otimes t^n]]. \end{aligned}$$

Hence \check{T} is a Lie triple system. \square

For convenience, we write $xt(xt^3)$ in place of $x \otimes t(x \otimes t^3)$.

If U is a subspace of T such that $T = U \oplus [T, T, T]$, then

$$\check{T} = Tt + Tt^3 = Tt + Ut^3 + [T, T, T]t^3,$$

Now we define a map $\varphi : \text{QDer}(T) \rightarrow \text{End}(\check{T})$ satisfying

$$\varphi(D)(at + ut^2 + bt^3) = D(a)t + D'(b)t^3,$$

where $D \in \text{QDer}(T)$, and D' is in Eq.(1.2), $a \in T, u \in U, b \in [T, T, T]$.

Proposition 4.2 T, \check{T}, φ are as defined above. Then

- (1) φ is injective and $\varphi(D)$ does not depend on the choice of D' .
- (2) $\varphi(\text{QDer}(T)) \subseteq \text{Der}(\check{T})$.

Proof. (1) If $\varphi(D_1) = \varphi(D_2)$, then for all $a \in T, b \in [T, T, T]$ and $u \in U$, we have

$$\varphi(D_1)(at + ut^3 + bt^3) = \varphi(D_2)(at + ut^3 + bt^3),$$

that is

$$D_1(a)t + D_2'(b)t^3 = D_2(a)t + D_2'(b)t^3,$$

so $D_1(a) = D_2(a)$. Hence $D_1 = D_2$, and φ is injective.

Suppose that there exists D'' such that

$$\varphi(D)(at + ut^3 + bt^3) = D(a)t + D''(b)t^3,$$

and

$$[D(x), y, z] + [x, D(y), z] + [x, y, D(z)] = D''([x, y, z]),$$

then we have

$$D'([x, y, z]) = D''([x, y, z]),$$

thus $D'(b) = D''(b)$. Hence

$$\varphi(D)(at + ut^3 + bt^3) = D(a)t + D'(b)t^3 = D(a)t + D''(b)t^3,$$

which implies $\varphi(D)$ is determined by D .

(2) We have $[xt^i, yt^j, zt^k] = [x, y, z]t^{i+j+k} = 0$, for all $i + j + k \geq 4$. Thus, to show $\varphi(D) \in \text{Der}(\check{T})$, we need only to check the validness of the following equation

$$\varphi(D)([xt, yt, zt]) = [\varphi(D)(xt), yt, zt] + [xt, \varphi(D)(yt), zt] + [xt, yt, \varphi(D)(zt)].$$

For all $x, y, z \in T$, we have

$$\begin{aligned} \varphi(D)([xt, yt, zt]) &= \varphi(D)([x, y, z]t^3) = D'([x, y, z])t^3 \\ &= ([D(x), y, z] + [x, D(y), z] + [x, y, D(z)])t^3 \\ &= [D(x)t, yt, zt] + [xt, D(y)t, zt] + [xt, yt, D(z)t] \\ &= [\varphi(D)(xt), yt, zt] + [xt, \varphi(D)(yt), zt] + [xt, yt, \varphi(D)(zt)]. \end{aligned}$$

Therefore, for all $D \in \text{QDer}(T)$, we have $\varphi(D) \in \text{Der}(\check{T})$. □

Proposition 4.3 Let T be a Lie triple system. $Z(T) = \{0\}$ and \check{T} , φ are as defined above. Then $\text{Der}(\check{T}) = \varphi(\text{QDer}(T)) \dot{+} \text{ZDer}(\check{T})$.

Proof. Since $Z(T) = \{0\}$, we have $Z(\check{T}) = Tt^3$. For all $g \in \text{Der}(\check{T})$, we have $g(Z(\check{T})) \subseteq Z(\check{T})$, hence $g(Ut^3) \subseteq g(Z(\check{T})) \subseteq Z(\check{T}) = Tt^3$. Now we define a map $f : Tt + Ut^3 + [T, T, T]t^3 \rightarrow Tt^3$ by

$$f(x) = \begin{cases} g(x) \cap Tt^3, & x \in Tt; \\ g(x), & x \in Ut^3; \\ 0, & x \in [T, T, T]t^3. \end{cases}$$

It is clear that f is linear. Note that

$$f([\check{T}, \check{T}, \check{T}]) = f([T, T, T]t^3) = 0,$$

$$[f(\check{T}), \check{T}, \check{T}] \subseteq [Tt^3, Tt + Tt^3, Tt + Tt^3] = 0,$$

hence $f \in \text{ZDer}(\check{T})$. Since

$$(g - f)(Tt) = g(Tt) - g(Tt) \cap Tt^3 = g(Tt) - Tt^3 \subseteq Tt, \quad (g - f)(Ut^3) = 0,$$

and

$$(g - f)([T, T, T]t^3) = g([\check{T}, \check{T}, \check{T}]) \subseteq [\check{T}, \check{T}, \check{T}] = [T, T, T]t^3,$$

there exist $D, D' \in \text{End}(T)$ such that for all $a \in T, b \in [T, T, T]$,

$$(g - f)(at) = D(a)t, \quad (g - f)(bt^3) = D'(b)t^3.$$

Since $(g - f) \in \text{Der}(\check{T})$ and by the definition of $\text{Der}(\check{T})$, we have

$$[(g - f)(a_1t), a_2t, a_3t] + [a_1t, (g - f)(a_2t), a_3t] + [a_1t, a_2t, (g - f)(a_3t)] = (g - f)([a_1t, a_2t, a_3t]),$$

for all $a_1, a_2, a_3 \in T$. Hence

$$[D(a_1), a_2, a_3] + [a_1, D(a_2), a_3] + [a_1, a_2, D(a_3)] = D'([a_1, a_2, a_3]).$$

Thus $D \in \text{QDer}(T)$. Therefore, $g - f = \varphi(D) \in \varphi(\text{QDer}(T))$, so $\text{Der}(\check{T}) \subseteq \varphi(\text{QDer}(T)) + \text{ZDer}(\check{T})$. By Proposition 4.2 (2) we have $\text{Der}(\check{T}) = \varphi(\text{QDer}(T)) + \text{ZDer}(\check{T})$.

For all $f \in \varphi(\text{QDer}(T)) \cap \text{ZDer}(\check{T})$, there exists an element $D \in \text{QDer}(T)$ such that $f = \varphi(D)$. Then

$$f(at + ut^3 + bt^3) = \varphi(D)(at + ut^3 + bt^3) = D(a)t + D'(b)t^3,$$

for all $a \in T, b \in [T, T, T]$.

On the other hand, since $f \in \text{ZDer}(\check{T})$, we have

$$f(at + bt^3 + ut^3) \in \text{Z}(\check{T}) = Tt^3.$$

That is to say, $D(a) = 0$, for all $a \in T$ and so $D = 0$. Hence $f = 0$.

Therefore $\text{Der}(\check{T}) = \varphi(\text{QDer}(T)) + \text{ZDer}(\check{T})$ as desired. \square

References

- [1] Benoist, Y. (1988). La partie semi-simple de l'algèbre des dérivations d'une algèbre de Lie nilpotente. C.R. Acad. Sci. Paris. 307:901-904.
- [2] Chen L. Y., Ma Y., Ni L. (2013). Generalized Derivations of Lie Color Algebras. Results Math. 63:923-936.
- [3] Fialkow, L. A. (1980). Generalied derivations. Topics in Modern Operator Theory. 95-103. Operator Theory: Adv. Appl., 2, Birkhäuser, Basel-Boston, Mass., 1981.

- [4] Filippov, V. T. (2000). On δ -derivations of prime alternative and Mal'tsev algebras. (Russian) *Algebra Log.* 39:618-625.
- [5] Filippov, V. T. (1998). On δ -derivations of Lie algebras. (Russian). *Sibirsk. Mat. Zh.* 39:1409-1422.
- [6] Hopkins, N. C. (1996). Generalized derivations of nonassociative algebras. *Nova J. Math. Game Theory Algebra.* 5:215-224.
- [7] Hopkins, N. C. (1985). Some structure theory for a class of triple systems. *Trans, Amer. Math. Soc.* 289:203-212.
- [8] Jacobson N. (1949). Lie and Jordan triple Systems. *Amer. J. Math. Soc.* 71:149-170.
- [9] Jimenez-Gestal, C., Perez-Izquierdo, J. M. (2008). Ternary derivations of finite-dimensional real division algebras. *Linear Algebra Appl.* 428:2192-2219.
- [10] Kamiya N., Okubo S. (1997). On triple systems and Yang-Baxter equations//*Proceedings of the Seventh International Colloquium on Differential Equations.* Utrecht:VSP. 189-196.
- [11] Leger, G. F., Luks, E. M. (2000). Generalized derivations of Lie algebras. *J. Algebra.* 228:165-203.
- [12] Lev S., Larissa S. and Ivan S. (2006). *Non-Associative Algebra and Its Applications.* CRC Press, U.S.A.
- [13] Lin, J. (2010). Centroid of Lie triple systems. *Acta Scientiarum Naturalium Universitatis Nankaiensis.* 43:98-104.
- [14] Lister, William G. (1952). A structure theory of Lie Triple systems. *Trans, Amer. Math. Soc.* 72:217-242.
- [15] Melville, D. J. (1992). Centroids of nilpotent Lie algebras. *Comm. Algebra.* 20:3649-3682.
- [16] Ma, Y., Chen, L. Y., Lin, J. (2014). Systems of quotients of Lie triple systems. *Comm. Algebra.* 42:3339-3349.
- [17] Shi, Y. Q., Meng, D. J. (2002). On derivations and automorphism group of Lie triple systems. *Acta Scientiarum Naturalium Universitatis Nankaiensis.* 35:32-37.
- [18] Vinberg, È. B. (1989). Generalized derivations of algebras. *Algebra and analysis.* 185-188. *Amer. Math. Soc. Transl. Ser. 2, 163, Amer. Math. Soc., Providence, RI, 1995.*
- [19] Zhang, R. X., Zhang, Y. Z. (2010). Generalized derivations of Lie superalgebras. *Comm. Algebra.* 38:3737-3751.